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A classification of special points of quasilattices in two dimensions

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Abstract. A complete classification of the special points of octagonal, decagonal and dodecagonal quasilattices in two dimensions (2D) is presented. They are obtained by projecting the special points of the starting n-gonal lattice ($n = 8$, 10 or 12) in 4D onto a 2D subspace. A set of equivalent special points of a quasilattice are located on the centres of a single kind of tile in the quasiperiodic tiling associated with the quasilattice; the point symmetry of the special points is identical to that of the tiles. They form a quasilattice with the same point symmetry as that of the original quasilattice. The new quasilattice is of a 'Bravais type' or 'non-Bravais type' according to whether the point symmetry group of the special points is identical to the full point symmetry group of the quasilattice or its true subgroup, respectively.

1. Introduction

A point $\boldsymbol{x}$ in a unit cell of a periodic lattice is called a special point if its point symmetry, $\mathrm{G}(\boldsymbol{x})$, has only a fixed point but does not have a fixed line nor a fixed plane. $\mathrm{G}(\boldsymbol{x})$ is isomorphic to a subgroup of the point symmetry group, G, of the lattice and the number of equivalent special points in the unit cell is given by $d(\boldsymbol{x}) = |\mathrm{G}|/|\mathrm{G}(\boldsymbol{x})|$, where the symbol $|*|$ stands for the order of the set $*$. A special point is nothing but a Wyckoff position without variable parameters in crystallographer's terminology (Wondratschek 1987).

The special points are also defined in the reciprocal space with respect to the reciprocal lattice. They are important in the band theory of solids because a special point has a higher point symmetry than its neighbouring points. The dispersion function of electrons is stationary at the special points.

For example, the point symmetry group of $\mathrm{Fm\bar{3}m/O_h^5}$, the face centred cubic lattice in the three dimensions (3D), is $\mathrm{O_h}$ and there exist four kinds of special points, Γ, H, P and N, where the symbols in the band theory are used. The point symmetries of these special points are $\mathrm{O_h}$, $\mathrm{O_h}$, T_d and D_{2h}, respectively. It follows that $d(\Gamma) = d(H) = 1$, $d(P) = 2$ and $d(N) = 6$. A representative special point in each class is given by $[000]$, $[100]$, $[\frac{1}{2}\frac{1}{2}\frac{1}{2}]$ or $[\frac{1}{2}\frac{1}{2}0]$, respectively.

In a crystal with a simple chemical formula, the special points of the relevant periodic lattice are probable positions to be occupied by the atoms. For example, sodium atoms occupy the Γ positions of the face centred cubic lattice and chlorine atoms occupy H positions in rock salt (NaCl). On the other hand, calcium atoms occupy Γ positions and fluorine atoms P positions in the case of the fluorite ($\mathrm{CaF_2}$).

Recently, the present author and his collaborator introduced special points in the reciprocal space of an icosahedral quasilattice and showed that the quasidispersion relation of electrons is stationary at these points (Niizeki and Akamatsu 1989). The

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special points are given by projections of the special points of a 6D simple hypercubic lattice onto the real (3D) reciprocal space.

In this paper, we shall introduce special points of a quasilattice in the real space and present a complete classification of them for the cases of octagonal, decagonal and dodecagonal quasilattices. If a quasiperiodic tiling is associated with a quasilattice in 2D, the centres of a single kind of tiles form a class of equivalent special points whose point symmetry is identical to that of the tiles. The special points of a quasilattice will be useful when decorating the quasilattice with atoms to represent a real quasicrystal.

In the case of a quasilattice there exists at most one point with a global (exact) point symmetry. Therefore, special points of a quasilattice cannot be defined in the same way as those of a periodic lattice. However, a quasilattice is obtained by the cut-and-projection method from a higher-dimensional periodic lattice (Elser 1986, Katz and Duneau 1986, Janssen 1986). Accordingly, special points of a quasilattice may be defined by the cut-and-projection method from the special points of the starting periodic lattice.

In § 2, we will summarise the definition and the properties of an n-gonal lattice (Niizeki 1989a). In § 3, we will establish a systematic method of enumerating all the special points of an n-gonal lattice and, in § 4, list the special points of octagonal, decagonal and dodecagonal lattices in 4D. In § 5, we will construct several 2D quasilattices by the cut-and-projection method and, in § 6, investigate the special points of these 2D quasilattices. In § 7, we will classify 2D quasilattices with global point symmetries. In the final section, § 8, we will present several remarks on the present work.

The reader who is interested only in the results but not in the mathematical details of their derivations can skip § 3.

2. The octagonal, decagonal and dodecagonal lattices in 4D

We shall identify the 2D Euclidean space E with the complex plane (Gauss–Argan plane) $\boldsymbol{C}$; $x=(x_1, x_2)\in \mathrm{E}_2 \leftrightarrow z=x_1+\mathrm{i}x_2\in \boldsymbol{C}$. Let $\zeta=\exp(2\pi\mathrm{i}/n)$ with $n=8$, 10 or 12. Then ζ is a primitive root of the n-cyclotomic equation, $x^n-1=0$. The n unit vectors, $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$, represent the vertex vectors of the unit regular n-gon Π_n centred on the origin. The set of complex numbers, $\mathrm{C}_n=\{1, \zeta, \zeta^2, \ldots, \zeta^{n-1}\}$, acts multiplicatively on $\boldsymbol{C}$ as a cyclic point symmetry group of $E_2 \simeq \boldsymbol{C}$; ζ acts as a rotation by $2\pi/n$. On the other hand, the complex conjugate operation which changes $z\in\boldsymbol{C}$ to $\bar{z}$ acts as a mirror (or reflection) with respect to the real axis of $\boldsymbol{C}$. If the complex conjugate operation is added to C_n, the resulting group is D_n, the dihedral point symmetry group of the plane. D_n is also the point symmetry group of Π_n. Note that $|\mathrm{D}_n|=2n$.

Of the n vectors, $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$, only four are linearly independent of each other over $\boldsymbol{Z}$, the integral domain of real integers, because ζ satisfies a quartic equation $P_n(x)=0$ with $P_8(x)=1+x^4$, $P_{10}(x)=1-x+x^2-x^3+x^4$ and $P_{12}(x)=1-x^2+x^4$. The four solutions of the equation are all primitive roots of the n-cyclotomic equation. They are given by ζ, ζ', $\bar{\zeta}$ and $\bar{\zeta}'$ with $\zeta'=-\zeta$ for $n=8$ or 12 and $\zeta'=\zeta^3$ for $n=10$. Note that the series, $1, \zeta', (\zeta')^2, \ldots, (\zeta')^{n-1}$, is a mere permutation of $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$.

The set of algebraic integers, $\boldsymbol{Z}(\zeta)=\{n_0+n_1\zeta+n_2\zeta^2+n_3\zeta^3 \mid n_i\in\boldsymbol{Z}\}$, generated by ζ is an integral domain. Note that $\boldsymbol{Z}(\zeta)=\boldsymbol{Z}(\zeta')$. The conjugate (not the complex conjugate) of $\nu\in\boldsymbol{Z}(\zeta)$ is defined by $\nu'=\nu|_{\zeta\to\zeta'}$ and its norm by $N(\nu)=|\nu\nu'|^2$. $N(\nu)$ is a positive integer unless $\nu=0$. Note that $(\nu')'=\nu$ (or $\bar{\nu}$ for $n=10$) and $N(\nu)=N(\nu')$.

$\tau \in \boldsymbol{Z}(\zeta)$ is called a unit of $\boldsymbol{Z}(\zeta)$ if $N(\tau)=1$, or equivalently, if $\tau^{-1} \in \boldsymbol{Z}(\zeta)$. All the units of $\boldsymbol{Z}(\zeta)$ form a multiplicative Abelian group generated by ζ and a fundamental unit τ_0. τ_0 is given by $1+\sqrt{2}$ $(=1+\zeta+\zeta^{-1})$, $(1+\sqrt{5})/2$ $(=\zeta+\zeta^{-1})$ or $1+\zeta$ for $n=8$, 10 or 12, respectively.

Let $\varepsilon_i = {}^t(\zeta^i, (\zeta')^i)$, $i=0, \ldots, (n-1)$, be 2D complex vectors, where the left superscript t stands for the transpose operation. Then, they are identified with 4D real vectors in E_4 $(\simeq \boldsymbol{R}^4 \simeq C^2)$. The first four, ε_0, ε_1, ε_2 and ε_3, of ε_i are linearly independent and the remaining $n-4$ are given as linear combinations of the four with integer coefficients. The 4D n-gonal lattice is defined by $L_n = \{n_0\varepsilon_0 + \ldots + n_3\varepsilon_3 \mid n_i \in \boldsymbol{Z}\}$ (Niizeki 1989a).

$E_4 \simeq \boldsymbol{C}^2$ is divided naturally into two 2D subspaces as $E_2 \oplus E_2 (\simeq \boldsymbol{C} \oplus \boldsymbol{C})$; the former is called the external space and the latter the internal one. We shall denote the projectors which project E_4 onto the two subspaces as π and π', respectively; $\boldsymbol{x} = x_0\varepsilon_0 + x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 \in E_4$ is projected as $\pi(\boldsymbol{x}) = x_0 + x_1\zeta + x_2\zeta^2 + x_3\zeta^3$ and $\pi'(\boldsymbol{x}) = x_0 + x_1\zeta' + x_2(\zeta')^2 + x_3(\zeta')^3$. It follows that $\pi(L_n) = \pi'(L_n) = \boldsymbol{Z}(\zeta)$. We will sometimes denote the 4D vector $\boldsymbol{x} = x_0\varepsilon_0 + \ldots + x_3\varepsilon_3$ as $\boldsymbol{x} = [x_0x_1x_2x_3]$. x_i are called indices of $\boldsymbol{x}$.

Let $z = {}^t(z_1, z_2) \in \boldsymbol{C}^2 (\simeq E_4)$. Then, an orthogonal transformation ρ of E_4 is defined by $\rho z = {}^t(\zeta z_1, \zeta' z_2)$. The order of ρ is equal to n; $\rho^n = 1$. ρ leaves L_n invariant; $\rho L_n = L_n$. Another orthogonal transformation σ of E_4 is defined by $\sigma z = (\bar{z}_1, \bar{z}_2)$. It has the properties, $\sigma^2 = 1$, $\sigma\rho\sigma^{-1} = \rho^{-1}$ and $\sigma L_n = L_n$. ρ and σ generate a group D'_n which is isomorphic to D_n; $\mathrm{D}'_n = \pi^{-1}(\mathrm{D}_n)$. D'_n leaves each subspace in $\boldsymbol{C} \oplus \boldsymbol{C} (= \boldsymbol{C}^2 \simeq E_4)$ invariant. D'_n is the maximal point symmetry group of L_n having this property. In what follows, we shall frequently denote D'_n simply as D_n.

We should mention here the arbitrariness in indexing lattice vectors of L_n. Let $\tau \in \boldsymbol{Z}(\zeta)$ be a unit. Then, the 4D lattice generated by $\tilde{\varepsilon}_i = {}^t(\zeta^i/\tau, (\zeta')^i/\tau')$, $i = 0, \ldots, 4$, is identical to L_n because there exists a unimodular matrix K such that $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3) = (\tilde{\varepsilon}_0, \tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \tilde{\varepsilon}_3)K$. The $\tilde{\varepsilon}_i$ are transformed by D_n linearly among themselves in the same way as the ε_i are. Therefore, there are no reasons why we should not index $\boldsymbol{x} \in E_4$ by the new basis vectors $\tilde{\varepsilon}_i$. We shall denote the new index with a tilde as $[\ldots]^{\sim}$. Thus, $\boldsymbol{x}$ is indexed in two ways as $\boldsymbol{x} = [x_0x_1x_2x_3] = [\tilde{x}_0\tilde{x}_1\tilde{x}_2\tilde{x}_3]^{\sim}$. The two sets of indices are related to each other by ${}^t(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = K\,{}^t(x_0, x_1, x_2, x_3)$. The arbitrariness in indexing L_n is closely related with the self-similarity of an n-gonal quasilattice.

A symmorphic space group in 4D is defined in terms of D_n and the translational group L_n. We shall denote it by $\mathscr{G}_n$. The point symmetry group of $\boldsymbol{x} \in E_4$ is defined by $\mathrm{G}(\boldsymbol{x}) = \{\alpha \mid \alpha \in \mathrm{G}$ and $\alpha\boldsymbol{x} - \boldsymbol{x} \in L_n\}$ with $\mathrm{G} = \mathrm{D}_n$. Note that $\alpha\boldsymbol{x} - \boldsymbol{x} \in L_n$ is equivalent to $\alpha\boldsymbol{x} \equiv \boldsymbol{x} \bmod L_n$. $\mathrm{G}(\boldsymbol{x})$ is a subgroup of G.

In the case of $n=8$, we find that $\varepsilon_i \cdot \varepsilon_j = 2\delta_{i,j}$. Therefore, L_8 is a 4D simple hypercubic lattice with the lattice constant $\sqrt{2}$. The full point symmetry group of L_8 is $\Omega(4)$, the 4D hyperoctahedral point group, whose order is equal to 384. However, only the subgroup D_8 of $\Omega(4)$ is relevant to the octagonal quasilattice to be derived from L_8.

In the case of $n=12$, ε_0 and ε_2 are orthogonal to ε_1 and ε_3 and L_{12} is represented as a direct product of two 2D lattices; $L_{12} = L_6 \times L'_6$ where L_6 (or L'_6) is a 2D hexagonal lattice generated by ε_0 and ε_2 (or ε_1 and ε_3).

3. The special points of an n-gonal quasilattice I: the general theory

Let us denote $\boldsymbol{x} + L_n (= \{\boldsymbol{x} + \boldsymbol{l} \mid \boldsymbol{l} \in L_n\})$ by $L_n(\boldsymbol{x})$, i.e. $L_n(\boldsymbol{x})$ is a simple translate of L_n. If $\boldsymbol{x}$ is a special point of L_n then $L_n(\boldsymbol{x})$ represents the set of all the special points equivalent to $\boldsymbol{x}$ with respect to the translational group L_n. Note that $\alpha L_n(\boldsymbol{x}) = L_n(\boldsymbol{x})$

for $\alpha \in \mathrm{G}(\boldsymbol{x})$. On the other hand, we shall denote by $L_n[\boldsymbol{x}]$ the set of all the equivalent special points to $\boldsymbol{x}$ with respect to the space group g_n. Since $\alpha L_n[\boldsymbol{x}] = L_n[\boldsymbol{x}]\ \forall \alpha \in \mathrm{D}_n$, $L_n[\boldsymbol{x}]$ is an n-gonal lattice composed of $d(\boldsymbol{x})(=|\mathrm{G}|/|\mathrm{G}(\boldsymbol{x})|)$ sublattices which are translationally congruent with L_n. It is a Bravais-type n-gonal lattice only when $d(\boldsymbol{x}) = 1$ (for a non-Bravais-type n-gonal lattice, see Niizeki 1989b).

The point symmetry group of a special point of L_n is equal to D_m or C_m where m is a divisor of n, but the case $m = 1$ is eliminated. The number of equivalent special points in a unit cell of L_n is given by $d = n/m$ if the relevant point group is D_m. d is called the order of the special points.

All the lattice points in L_n form an important class of special points, which is denoted by Γ following the convention of the band theory. Note that $\mathrm{G}(\Gamma) = \mathrm{D}_n$ and $d(\Gamma) = 1$.

Let $m(\neq 1)$ be a diviser of n and assume that $\boldsymbol{x} \in E_4$ is a special point such that $\mathrm{G}(\boldsymbol{x}) \supset \mathrm{C}_m$. Then, we obtain $\rho^k \boldsymbol{x} \equiv \boldsymbol{x} \bmod L_n$ with $k = n/m(<n)$ is another divisor of n. Projecting this condition onto the external space yields that $\zeta^k z \equiv z \bmod \boldsymbol{Z}(\zeta)$ with $z = \pi(\boldsymbol{x})$. We assume that $\boldsymbol{x} \notin L_n$, i.e. $\boldsymbol{x}$ is not a Γ point. Then, $z \in \boldsymbol{Z}(\zeta)$, so that $\mu = 1 - \zeta^k$ is not a unit of $\boldsymbol{Z}(\zeta)$. It follows that $z \in J$ where $J = \beta \boldsymbol{Z}(\zeta)$ with $\beta = 1/(1-\zeta^k)$ is a fractional ideal. Note that $\boldsymbol{Z}(\zeta) \subset J$ because $1 = \beta\mu \in J$. Note also that J is self-conjugate, $J = \bar{J}$, because $\bar{\beta} = -\zeta^k \beta$.

From $z = \pi(\boldsymbol{x}) \in J$, we can conclude that $\boldsymbol{x} \in M$ with $M = \pi^{-1}(J)$ (the domain of π^{-1} is extended naturally from $\boldsymbol{Z}(\zeta)$ to $\boldsymbol{Q}(\zeta)$). M is a 4D lattice generated by the basis vectors, $\varepsilon'_i = {}^{t}(\beta\zeta^i, \beta'(\zeta')^i)$, $i = 0, \ldots, 3$, with $\beta'(=1/\mu' = 1/[1-(\zeta')^k])$ being the conjugate of β. M is an n-gonal lattice and L_n is a D_n-superlattice of M (Niizeki 1989b) because J is a self-conjugate ideal of $\boldsymbol{Z}(\zeta)$ and $J \supset \boldsymbol{Z}(\zeta)$. We shall call M a miniature lattice of L_n generated by β and denote it as $M = L_n\{\beta\}$. The generator of M is determined up to a multiplicative factor being a unit. M is similar to L_n only when it has a generator β such that β', the conjugate of β, satisfies $|\beta| = |\beta'|$; $|\beta|$ is then equal to the ratio between the lattice constants of M and L_n.

We can show that $\mathrm{G}(\boldsymbol{x}) \supset \mathrm{D}_m\ \forall \boldsymbol{x} \in M$ because J is self-conjugate. Therefore, all the lattice points of M are special points of L_n. Moreover, we can conclude that L_n has no special points whose point symmetries are cyclic groups.

Let $\Lambda = M/L_n$, the residue class module. Then, we obtain that $\Lambda \simeq J/\boldsymbol{Z}(\zeta) \simeq \boldsymbol{Z}(\zeta)/(\mu\boldsymbol{Z}(\zeta))$, which is a residue class ring. It follows that $q \equiv |\Lambda| = N(\mu)$ and M is divided into q sublattices which are translationally equivalent to L_n.

If $m = 2$ then $\mu = 2$, $\beta = \frac{1}{2}$, $q = 16$ and $M = L_n\{\frac{1}{2}\}$, which we may call a half lattice of L_n. On the other hand, we obtain $\mathrm{C}_2 = \{1, I\}$ with $I = \rho^{n/2}$ being the inversion operation; $\boldsymbol{x} \in E_4 \to I\boldsymbol{x} = -\boldsymbol{x}$. We shall denote a special point of L_n as type I or II according to whether or not its point symmetry group includes I. A necessary and sufficient condition for $\boldsymbol{x} \in E_4$ to be a type-I special point of L_n is that $\boldsymbol{x} \in L_n\{\frac{1}{2}\}$. The following proposition is proved easily. Let $\boldsymbol{x}$ be a special point of L_n and assume that its point symmetry group is D_m with $m(\neq 1)$ a divisor of n. Then, m is even or odd according as the special point $\boldsymbol{x}$ is of type I or type II, respectively. A Γ point is of type I.

It follows that L_8 has no type-II special points. Moreover, the point symmetry group of a type-II special point of L_n is D_5 if $n = 10$, but D_3 if $n = 12$.

We now return to the miniature lattice $M = L_n\{\beta\}$ with $\beta = 1/(1-\zeta^k)$. The q sublattices of M are permuted among themselves by every operation in D_n. Therefore, they are grouped into a number of irreducible subsets with respect to D_n. This grouping is performed in a similar way as the one by which a similar problem is solved in Niizeki

(1989b). The union of all the sublattices in the ith subset is identical to $L_n[\boldsymbol{x}_i]$, where $\boldsymbol{x}_i$ is a representative point in the union. The number of the sublattices in $L_n[\boldsymbol{x}_i]$ is given by $d_i = |\mathrm{D}_n|/|\mathrm{G}(\boldsymbol{x}_i)|$. We can assume that $\boldsymbol{x}_0 = 0$, $d_0 = 1$ and $L_n[\boldsymbol{x}_0] = L_n$. Thus, we obtain $M = L_n \cup L_n[\boldsymbol{x}_1] \cup \ldots \cup L_n[\boldsymbol{x}_{s-1}]$ and $q = 1 + d_1 + \ldots + d_{s-1}$, where s is the total number of the subsets. That is, M is divided into s equivalence classes of the special points of L_n.

If $\mu = 1 - \zeta^k$ is a prime number in $\boldsymbol{Z}(\zeta)$ then, $q = p^j$ $(j \geqslant 1)$ with p being a prime integer and Λ is a finite field. It can be shown in this case that $\mathrm{G}(\boldsymbol{x}_i)$ is equal to D_m for all i, so that $d_1 = d_2 = \ldots = d_{s-1} = k$ and $s - 1 = (q-1)/k$.

If $\exists \nu \in \boldsymbol{Z}(\zeta)$ such that $\mu\nu = p$, as in the case that μ is a prime number then, $J \subset p^{-1}\boldsymbol{Z}(\zeta)$ and $M \subset p^{-1}L_n$, so that $x \in M$ is indexed as $[n_0 n_1 n_2 n_3]/p$ with n_i being integers. An element of M/L_n can be indexed in this way, but with $0 \leqslant n_i \leqslant p-1$ or $-r \leqslant n_i \leqslant r$ with $r = (p-1)/2$, the latter part of which does not apply to the case $p = 2$.

If $\mu = 1 - \zeta^k$ is not a prime in $\boldsymbol{Z}(\zeta)$, it has a non-trivial divisor $\mu' = 1 - \zeta^{k'}$ with k' being a diviser of k. Then, $L_n\{1/\mu'\} \subset L_n\{1/\mu\}$; the former is a D_n-superlattice of the latter and consists only of special points. On the contrary, $L_n\{1/\mu\}$ has no such superlattice if μ is a prime in $\boldsymbol{Z}(\zeta)$.

A set of translationally equivalent special points of L_n includes one and only one which belongs to a unit cell (a fundamental domain) of L_n. We can take it as being representative of the set. The Voronoi cell (Wigner–Seitz cell), V_n, of the origin of L_n is a symmetric unit cell. Here, the points on one half of the boundary of V_n are included in V_n but those in the other half are not. V_n is a 4D polytope whose 3D hypersurfaces bisect lattice vectors representing neighbouring lattice points to the origin. The number, F_n, of 3D hypersurfaces of V_n satisfies $F_n \geqslant n$ with n equal to the coordination number of L_n; we obtain, in fact, that $F_8 = 8$, $F_{10} = 30$ and $F_{12} = 12$.

Let $\boldsymbol{x}$ be an interior point of V_n. Then we obtain $\alpha\boldsymbol{x} = \boldsymbol{x}\ \forall\alpha \in \mathrm{G}(\boldsymbol{x})$, so that $\{c\boldsymbol{x} \mid c \in \boldsymbol{R}\} \subset E_4$ is a fixed set of $\mathrm{G}(\boldsymbol{x})$. Consequently, $\boldsymbol{x}$ can be a special point only when $x = 0$. Thus, other special points in V_n than the Γ point are all located on the boundary of V_n. They must be on the vertices of V_n or on the centres of j-dimensional $(1 \leqslant j \leqslant 3)$ surfaces of V_n, which can be proved in a similar way to the case of an interior point of V_n.

4. The special points of an *n*-gonal quasilattice II: applications

4.1. *The octagonal lattice*

V_8 is a 4D hypercube which has eight 3D hypersurfaces and sixteen vertices. The sixteen type-I special points in V_8 are classified into six classes as given in table 1. The centre of a hypersurface, which is a cube, belongs to X (more exactly, $L_8[X]$), a vertex to O, the middle point of an edge to R and a centre of a 2D surface, which is a square, to C or C'. C and C' would be equivalent if the point symmetry of L_8 were $\Omega(4)$.

Table 1. The special points of the octagonal lattice.

symbol	Γ	X	C	C'	R	O
point symmetry	D_8	D_2	D_2	D_4	D_2	D_8
order	1	4	4	2	4	1
representative	[0000]	$[\frac{1}{2}000]$	$[\frac{1}{2}\frac{1}{2}00]$	$[\frac{1}{2}0\frac{1}{2}0]$	$[0\frac{1}{2}\frac{1}{2}\frac{1}{2}]$	$[\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}]$

The two numbers $\mu_1 = 1 - \zeta$ and $\mu_2 = 1 - \zeta^2$ with $N(\mu_1) = 2$ and $N(\mu_2) = 4$ are divisors of $2(=1-\zeta^4)$ in $\boldsymbol{Z}(\zeta)$. The relevant three miniature lattices, $L_{8,1} = L_8\{1/\mu_1\}$, $L_{8,2} = L_8\{1/\mu_2\}$ and $L_8\{\frac{1}{2}\}$ with $1/\mu_1 = (1+\zeta+\zeta^2+\zeta^3)/2$ and $1/\mu_2 = (1+\zeta^2)/2$, are associated with special points with point symmetries D_8, D_4 and D_2, respectively, and satisfy the relationships $L_{8,1} \subset L_{8,2} \subset L_8\{\frac{1}{2}\}$. More precisely, $L_{8,1} = L_8 \cup L_8[O]$ and $L_{8,2} = L_{8,1} \cup L_8[C']$. $L_{8,2}$ is similar to L_8, while $L_{8,1}$ is identical to a 4D body centred hypercubic lattice.

L_8 does not have type-II special points, as noted previously.

4.2. *The decagonal lattice*

2 is a prime number of $\boldsymbol{Z}(\zeta)$ and the sixteen type-I special points of L_{10} are classified into four classes as given in table 2. Note that four X points other than the representative in the table are given by $[0\frac{1}{2}00]$, $[00\frac{1}{2}0]$, $[000\frac{1}{2}]$ and $[\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}]$; the last X point is translationally equivalent to $\frac{1}{2}\varepsilon_4$ because $\varepsilon_0 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4 = 0$. Similarly, the fourth C (or C') point is given by $[\frac{1}{2}\frac{1}{2}\frac{1}{2}0]$ (or $[\frac{1}{2}\frac{1}{2}0\frac{1}{2}]$).

Table 2. The special points of the decagonal lattice. The first four are of type I and the last two are of type II.

symbol	Γ	X	C	C'	P	P'
point symmetry	D_{10}	D_2	D_2	D_2	D_5	D_5
order	1	5	5	5	2	2
representative	[0000]	$[\frac{1}{2}000]$	$[\frac{1}{2}\frac{1}{2}00]$	$[\frac{1}{2}0\frac{1}{2}0]$	$[\frac{2}{5}\frac{1}{5}\frac{1}{5}\frac{2}{5}]$	$[\frac{1}{5}\frac{2}{5}\frac{2}{5}\frac{1}{5}]$

The type-II special points are obtained from $L_{10}\{\beta\}$ with $\beta = 1/(1-\zeta^2)$. $q = N(1-\zeta^2) = 5$ and $\Lambda \simeq \boldsymbol{Z}_5(=\boldsymbol{Z}/5\boldsymbol{Z})$. Using the equalities $\zeta^2\beta = (2+\zeta+\zeta^2+2\zeta^3)/5$ and $2\zeta^2\beta \equiv (-1+2\zeta+2\zeta^2-\zeta^3)/5 \bmod \boldsymbol{Z}(\zeta)$, we obtain $\Lambda = \{0, \boldsymbol{x}_1, -\boldsymbol{x}_1, \boldsymbol{x}_2, -\boldsymbol{x}_2\} \bmod L_{10}$ with $\boldsymbol{x}_1 = [2112]/5$ and $\boldsymbol{x}_2 = [\bar{1}22\bar{1}]/5$. $L_{10}\{\beta\}$ is divided into three sublattices as $L_{10} \cup L_{10}[\boldsymbol{x}_1] \cup L_{10}[\boldsymbol{x}_2]$ with $L_{10}[\boldsymbol{x}_i] = L_{10}(\boldsymbol{x}_i) \cup L_{10}(-\boldsymbol{x}_i)$, $i = 1, 2$. The last two of the three present two classes of type-II special points whose point symmetry groups are both D_5, as summarised in table 2. The two classes of special points were noticed by Janssen (1986).

V_{10} is a 4D polytope with thirty 3D hypersurfaces; ten of them bisect the lattice vectors of the nearest neighbours of the origin and the remaining twenty bisect those of the second neighbours. The centres of the former ten belong to X and those of the latter twenty to C and C'. P and P' are located on vertices of V_{10}. Note, however, that V_{10} has other vertices which are not special points of L_{10}.

4.3. *The dodecagonal lattice*

The sixteen type-I special points are classified into four classes, as given in table 3. Note that $L_{12}[\Gamma] \cup L_{12}[C'] = L_{12}\{1/\mu\}$, where $\mu = 1 - \zeta^3(=\sqrt{2}\exp(-\pi i/4))$ with $q = N(\mu) = 4$ being a prime divisor of 2. $L_{12}\{1/\mu\}$ is similar to L_{12}.

The type-II special points are obtained from $L_{12}\{\beta\}$ where $\beta = 1/(1-\zeta^4)$ with $q = N(1-\zeta^4) = 9$. $\Lambda \simeq \boldsymbol{Z}_3(i)(\simeq \boldsymbol{Z}_3 + \mathrm{i}\boldsymbol{Z}_3)$. The miniature lattice is similar to L_{12}. It includes two classes of type-II special points, as listed in table 3.

Table 3. The special points of the dodecagonal lattice. The first four are of type I and the last two are of type II.

symbol	Γ	X	C	C'	T	T'
point symmetry	D_{12}	D_2	D_2	D_4	D_3	D_3
order	1	6	6	3	4	4
representative	[0000]	$[\frac{1}{2}000]$	$[\frac{1}{2}\frac{1}{2}00]$	$[\frac{1}{2}00\frac{1}{2}]$	$[\frac{1}{3}0\frac{1}{3}0]$	$[\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3}]$

Since L_{12} is a direct product of two hexagonal lattices L_6 and L'_6 in 2D, V_{12} is a 4D hyperprism; $V_{12} = H \times H'$, where H and H' are regular hexagons representing the Voronoi cells of L_6 and L'_6. V_{12} has 36 ($=6\times 6$) vertices, which belong to T'. Its twelve 3D hypersurfaces are hexagonal prisms in 3D. One of them is given by $P_6 = H \times S$ with S being one of the six sides of H'. The centre of P_6 belongs to X. The centres of the two hexagonal sides (surfaces) of P_6 belong to T. On the other hand, three of the centres of the six square sides belong to C and the remaining three to C'.

We have noted an arbitrariness in indexing an n-gonal lattice. Then, do the symbols assigned to different classes of the special points have definite meanings? The answer is affirmative for Γ of L_n for all n, C, C' and O of L_8 and C' of L_{12} but is negative for other classes. For example, the triplet $\{X, C, C'\}$ of L_{10} have a common point symmetry group, D_2, and are permuted cyclically among themselves if the index scheme is changed. There are many doublets with similar properties: $\{X, R\}$ of L_8, $\{P, P'\}$ of L_{10} and $\{X, C\}$ and $\{T, T'\}$ of L_{12}. These results are consequences of the fact that different sublattices in a miniature lattice are permuted among themselves under a self-similarity transformation (Niizeki 1989b). Since the transformation preserves the symmetry, the members of a multiplet must have a common point symmetry group.

5. The octagonal, decagonal and dodecagonal quasilattices

We consider in this section only the Bravais-type quasilattices (Niizeki 1989a). An n-gonal ($n = 8$, 10 or 12) quasilattice in 2D is a set of points in the external space given by

$$Q_n(\phi, W) = \{\pi(\boldsymbol{l}) \,|\, \boldsymbol{l} \in L_n \text{ and } \pi'(\boldsymbol{l}) \in \phi + W\} \tag{1a}$$

$$= \left\{ \sum_{i=0}^{3} n_i \zeta^i \,\Big|\, n_i \in \boldsymbol{Z} \text{ and } \sum_{i=0}^{3} n_i (\zeta')^i \in \phi + W \right\} \tag{1b}$$

where ϕ is a complex number representing the phase vector and W a polygonal domain (in the internal space) with point symmetry D_n. The condition that $\pi'(\boldsymbol{l}) \in \phi + W$ is equivalent to the condition that $\boldsymbol{l}$ is included in the strip $S(\phi, W)$ obtained by thickening the external space with W and translating subsequently by ϕ along the internal space. The strip $S(\phi, W)$ is called regular if its boundary includes no lattice points of L_n, and is called singular otherwise.

The n-gonal quasilattice gives naturally a quasiperiodic tiling (QPT) of the plane in terms of a finite number of polygonal tiles (Niizeki 1989a). We will sometimes identify a quasilattice with the QPT associated with it.

The n-gonal quasilattice has a macroscopic point symmetry given by $G = D_n$. It also has a self-similarity which is characterised by a complex similarity ratio given by a unit of $\boldsymbol{Z}(\zeta)$.

The local isomorphism class to which $Q_n(\phi, W)$ belongs is determined by the window W and is independent of ϕ. Therefore, we will sometimes denote it simply as $Q_n(W)$.

We now investigate the three cases separately.

5.1. *The octagonal quasilattice*

The projection of a unit cell of L_8 onto the internal space is a regular octagon Π'_8, which is related to the 'unit octagon' Π_8 by $\Pi'_8 \equiv \pi'(V_8) = \gamma\Pi_8$ with $\gamma = (1+\zeta)/2$. The relevant QPT is composed of square tiles and rhombic tiles (Ishihara *et al* 1987, see also Niizeki 1989c).

5.2. *The decagonal quasilattice*

The decagonal quasilattice, $Q_{10}(\Pi_{10})$, gives rise to a QPT with four kinds of tiles, i.e. a regular pentagon, a thin rhombus, a crown and a pentagonal star (Penrose 1974, Niizeki 1989a) as shown in figure 1. This quasilattice is identical to the projection of an icosahedral quasilattice along a fivefold axis.

We consider a different choice of the window W. Let Π_5 be the unit regular pentagon whose vertices are given by 1, ζ^2, ζ^4, ζ^6 and ζ^8 and let $-\Pi_5$ be its inversion. Then, $\gamma\Pi_5 \cup (-\gamma\Pi_5)$ with $\gamma = 1+\zeta^3$ is a decagonal star Σ_{10} whose ten obtuse vertices coincide with the vertices of Π_{10}. Since $\Sigma_{10} \supset \Pi_{10}$, the decagonal quasilattice, $Q_{10}(\Sigma_{10})$, is obtained from $Q_{10}(\Pi_{10})$ by adding new lattice points to it. Each of the new points appears in such a way that a pentagonal tile of the old QPT is divided into a thick rhombus and a part of a pentagonal star. One (or two) of the parts and a crown (or

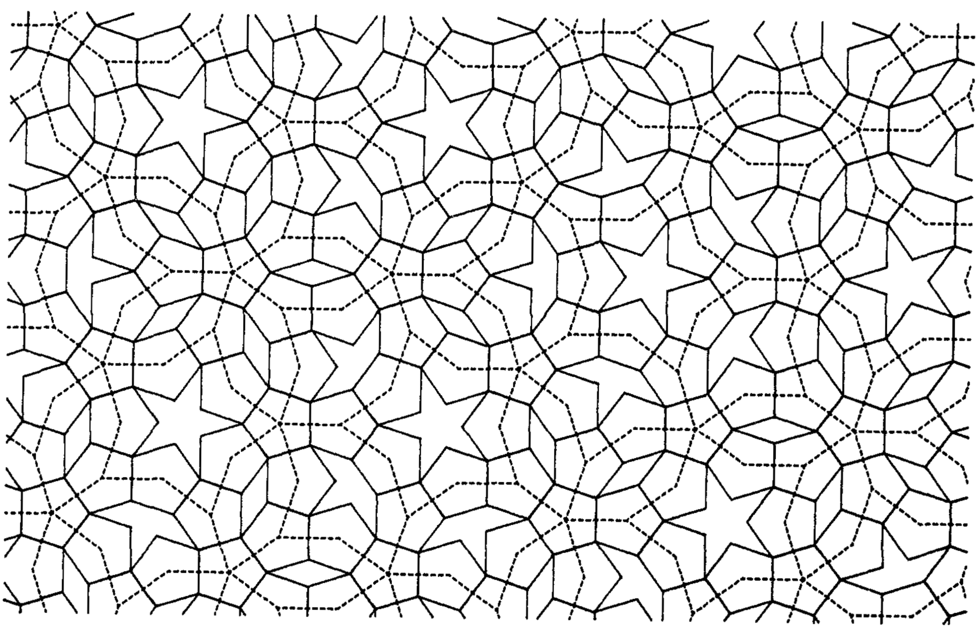

Figure 1. A Bravais-type decagonal quasilattice (solid lines) obtained with a window given by the unit regular decagon Π_{10}. The special points located on the centres of pentagonal tiles form a non-Bravais-type decagonal quasilattice (broken lines), which consists of two sublattices corresponding to the two possible orientations of the pentagons.

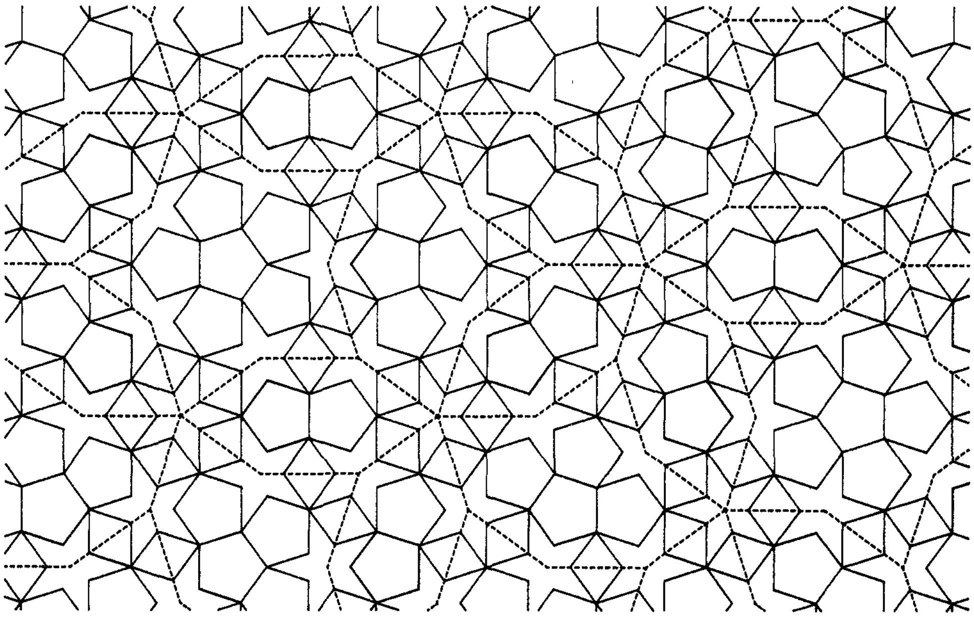

Figure 2. A Bravais-type decagonal quasilattice (solid lines) obtained with a window given by the decagonal star Σ_{10}. A pentagonal tile in figure 1 is divided into a thick rhombus and a part of a pentagonal star if and only if the centre of the tile is a two-pronged vertex of the 'dual quasilattice'. The special points located on the centres of pentagonal stars form a non-Bravais-type decagonal quasilattice (broken lines), which is similar with ratio τ to the one in figure 1.

a thin rhombus) of the old QPT are joined into a pentagonal star. Thus, the new QPT is composed of three kinds of tiles, namely, the pentagonal star, a thick rhombus and a regular pentagon, as shown in figure 2.

5.3. The dodecagonal quasilattice

The dodecagonal quasilattice, $Q_{12}(\Pi_{12})$, gives rise to a QPT with three kinds of tiles, i.e. an equilateral triangle, a square and a trigonal hexagon as shown in figure 3 (Niizeki and Mitani 1987). The dodecagonal quasilattice, $Q_{12}(\Pi'_{12})$, where $\Pi'_{12} \equiv \pi'(V_{12}) = \gamma\Pi_{12}$ with $\gamma = (1+\zeta)/\sqrt{3}$ is derived from $Q_{12}(\Pi_{12})$ by adding new lattice points to it. The new QPT is composed of three kinds of tiles, namely, the equilateral triangle, the square and the thin rhombus (Stampfli 1986, Niizeki and Mitani 1987).

6. Special points of n-gonal quasilattice in 2D

We now consider how we should define special points in the case of the n-gonal quasilattice, $Q_n(W)$, with $n = 8$, 10 or 12. Let $L_n[A]$ be the set of all the type-A special points of L_n. Then, $\pi(L_n[A])(=\{\pi(x) | x \in L_n[A]\})$ as well as $\pi(L_n)$ is a dense set in the external space. Therefore, it is not very useful to define all the points in $\pi(L_n[A])$ to be special points of $Q_n(\phi, W)$. We should cut $L_n[A]$ prior to projecting it. Then,

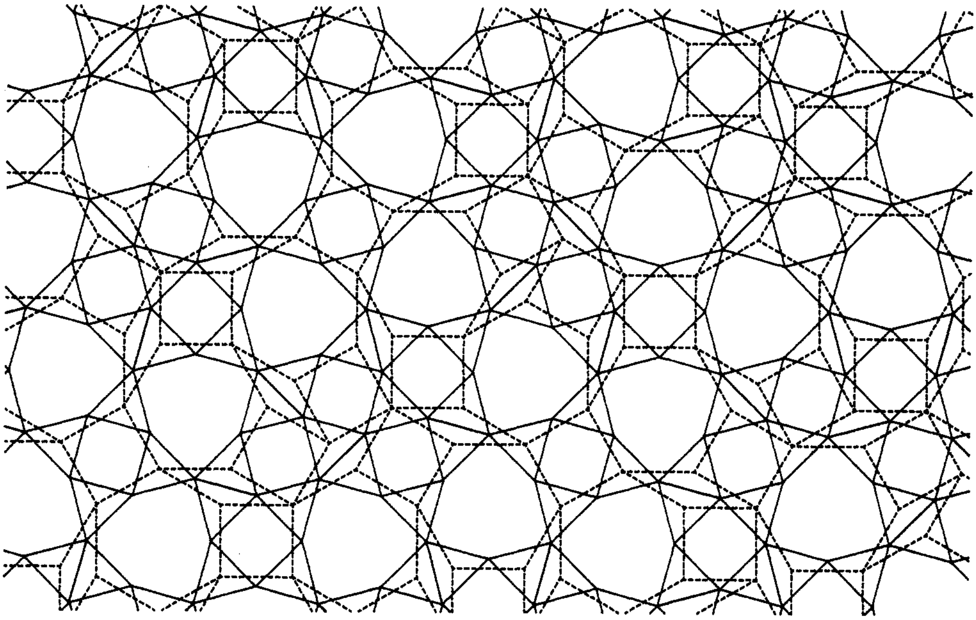

Figure 3. The solid lines represent a Bravais-type dodecagonal quasilattice $Q_{12}(\Pi_{12})$ and broken lines a non-Bravais-type quasilattice which is formed by the type T' special points located on the centres of equilateral triangles. The latter quasilattice consists of four sublattices corresponding to the four possible orientations of the triangles. The centres of squares (or trigonal hexagons) are the type C' (or T) special points.

by what criterion should we cut it? One obvious choice is to use the strip, $S(\phi, W)$. This choice is reasonable when $A = \Gamma$ because each lattice point in the quasilattice should be its Γ point, as in the case of a periodic lattice. This choice is, however, not necessarily the best choice in other cases, which we shall consider below.

We take as an example the decagonal quasilattice, $Q_{10}(\Pi_{10})$. The relevant QPT contains pentagonal tiles, which can take two orientations, as seen in figure 1. They are translationally congruent with $\pm\Pi'_5$, where $\Pi'_5 = \beta\Pi_5$ with $\beta = 1/(1-\zeta^2)$ $(= \exp(3\pi i/10)/2\sin(\pi/5))$. We shall call each orientation positive or negative, respectively.

In order that $z \in C$ is the centre of a positive pentagon, it is necessary that $z + \beta = \pi(\boldsymbol{l})\ \exists \boldsymbol{l} \in L_{10}$ because β represents a vertex of Π'_5. Since $\beta = \pi(\boldsymbol{x}_1)$ with $\boldsymbol{x}_1 =$ [2112]/5 being a representative of $L_{10}[P]$, we find that $z = \pi(\boldsymbol{x})$ with $\boldsymbol{x} = -x_1 + \boldsymbol{l} \in L_n(-\boldsymbol{x}_1)$. The condition, $\pi'(\boldsymbol{l}) \in \phi + W$ with $W = \Pi_{10}$, is rewritten with $\boldsymbol{x} = -\boldsymbol{x}_1 + \boldsymbol{l}$ as $\pi'(\boldsymbol{x}) \in \phi - \beta' + W$ with $\beta' = \pi'(\boldsymbol{x}_1)$ $(= 1/(1+\zeta) = \exp(-\pi i/10)/(2\sin(2\pi/5)))$ being the conjugate of β. By performing similar arguments for the other four vertices of Π'_5, we obtain that $\pi'(\boldsymbol{x}) \in \phi - \beta'(\zeta')^{2i} + W$ for $i = 1, \ldots, 4$. The five conditions can be cast into a single condition, $\pi'(\boldsymbol{x}) \in \phi + W'$, where W' is the common part of the five domains, $-\beta'(\zeta')^{2i} + W$, $i = 0, \ldots, 4$. It is a problem of elementary geometry to show that $W' = -\Pi''_5$ with $\Pi''_5 \equiv \beta'\Pi_5$, so that the set of the centres of all the positive pentagonal tiles in the QPT is given by $\{\pi(\boldsymbol{x}) \,|\, \boldsymbol{x} \in L_n(-\boldsymbol{x}_1)$ and $\pi'(\boldsymbol{x}) \in \phi - \Pi''_5\}$.

By means of a similar argument for the case of the negative pentagons, we arrive at the conclusion that the set of the centres of all the pentagonal tiles is given by

$$Q_{10,P}(\phi, \Pi''_5) = \{\pi(\boldsymbol{x}) \,|\, \boldsymbol{x} \in L_{10}[P] \text{ and } \pi'(\boldsymbol{x}) \in \phi \pm \Pi''_5\} \tag{2}$$

where the sign in $\pm\Pi''_5$ depends on which of the two sublattices of $L_{10}[P](=L_{10}(\boldsymbol{x}_1)\cup L_{10}(-\boldsymbol{x}_1))$ $\boldsymbol{x}$ belongs to.

$L_{10}[P]$ is a non-Bravais-type sublattice of the n-gonal lattice M, which is a miniature lattice of L_{10} introduced in § 4. It can be shown easily that $Q_{10,P}\equiv Q_{10,P}(\Pi''_5)$ is identical to the homopolar non-Bravais-type decagonal quasilattice constructed in Niizeki (1989b) as given in figure 1. It is also a sublattice of the Penrose lattice; it is obtained from the latter by discarding the lattice points with de Bruijin's indices 1 and 4 (de Bruijin 1981, Niizeki 1989b). This relationship between the pentagonal QPT and the Penrose lattice (tiling) is implicitly noticed by Henley (1986), though no proofs are presented.

By a similar argument, we can show that the set of the centres of all the pentagonal stars in the QPT is a non-Bravais-type decagonal quasilattice $Q_{10,P'}(\tau^{-3}\Pi''_5)$ derived from $L_{10}[P']$. The lattice points of this quasilattice coincide with the 'stars' S in the corresponding Penrose lattice mentioned above. It is similar to $Q_{10,P}$ with the ratio, $\tau^3(\tau=(1+\sqrt{5})/2)$.

On the other hand, the set of the middle points of all the bonds (the edges of the pentagonal tiles) in the QPT is another non-Bravais-type decagonal quasilattice derived from $L_{10}[X]$. The five sublattices of $L_{10}[X]$ correspond to the five orientations of the bonds. Similarly, the centres of the thin rhombi are special points derived from $L_{10}[C]$.

We consider next the special points of $Q_{10}(\Sigma_{10})$ given in figure 2. The set of the centres of all the pentagonal stars in it is a decagonal quasilattice $Q_{10,P'}=Q_{10,P'}(\tau^{-1}\Pi''_5)$, which is similar to $Q_{10,P}$. The union of the two quasilattices, $Q_{10,P}\cup Q_{10,P'}$, is identical to the Penrose lattice, as noted in Niizeki (1989b). Similarly, the centres of the thick rhombi in the QPT are special points of type C'; special points of this type are absent in $Q_{10}(\Pi_{10})$. Since the thin rhombi in $Q_{10}(\Pi_{10})$ are changed in $Q_{10}(\Sigma_{10})$ into pentagonal stars, all the type-C special points in $Q_{10}(\Pi_{10})$ disappear in $Q_{10}(\Sigma_{10})$. That is, whether $Q_{10}(W)$ has special points corresponding to a specified class or not is dependent on W.

We present several remarks on the special points of octagonal and dodecagonal quasilattices. The lattice points (or the middle points of the bonds) in $Q_8(\Pi'_8)$, $Q_{12}(\Pi_{12})$ and $Q_{12}(\Pi'_{12})$ are special points corresponding to Γ (or X). The centres of the polygonal tiles which have their own point symmetries are special points of the types with the same point symmetries; a rhombus and a square in $Q_8(\Pi'_8)$ (or $Q_{12}(\Pi'_{12})$) correspond to C and C', respectively, an equilateral triangle in the two dodecagonal quasilattices to T' and a trigonal hexagon in $Q_{12}(\Pi_{12})$ to T. On the contrary, special points corresponding to R and O are absent in $Q_8(\Pi'_8)$, C is absent in $Q_{12}(\Pi_{12})$ and T in $Q_{12}(\Pi'_{12})$. The final remark is that the quasilattice associated with the type T' special points in $Q_{12}(\Pi_{12})$ coincides, as shown in figure 3, with the non-Bravais-type dodecagonal quasilattice constructed in Niizeki (1988a), which can be proved rigorously in a similar way to the one in the case of decagonal quasilattices.

7. An application of the theory to a classification of n-gonal quasilattices with global point symmetries

Let $\boldsymbol{x}_0$ be a representative in $L_n[A]$ with A being a class of special points of L_n, and assume that $S(\phi, W)$ with $\phi=-\pi'(\boldsymbol{x}_0)$ is a regular strip. Then, the condition that $\pi'(\boldsymbol{l})\in\phi+W$ with $\boldsymbol{l}\in L_n$ is equivalent to $\pi'(\boldsymbol{x}_0+\boldsymbol{l})\in W$. Therefore, the shifted n-gonal quasilattice, $Q_n(A, W)=\pi(\boldsymbol{x}_0)+Q_n(\phi, W)$, is rewritten as

$$Q_n(A, W)=\{\pi(\boldsymbol{x})\,|\,\boldsymbol{x}\in L_n(\boldsymbol{x}_0) \text{ and } \pi'(\boldsymbol{x})\in W\}. \tag{3}$$

That is, $Q_n(A, W)$ is identical to the projection of a cut of $L_n(\boldsymbol{x}_0)$ with $S(0, W)$. Since $\alpha L_n(\boldsymbol{x}_0) = L_n(\boldsymbol{x}_0)$ and $\alpha S(0, W) = S(0, W)$ $\forall \alpha \in \mathrm{G}(\boldsymbol{x}_0)$, $Q_n(A, W)$ has $\mathrm{G}(\boldsymbol{x}_0)$ as its global (or exact) point symmetry group with respect to the origin.

In the case where $S(\phi, W)$ is singular, there occurs a frustration as to whether or not the lattice points on the boundary of the strip should be projected. This frustration is resolved by an infinitesimal change of ϕ. Unfortunately, the exact point symmetry is broken in this procedure, which one can say is a spontaneous symmetry breaking (de Bruijin 1981, Niizeki 1988b, 1989d). The ratio of the number of the lattice points breaking the symmetry is, however, infinitesimal, so that $\mathrm{G}(\boldsymbol{x}_0)$ is virtually the global point symmetry group of $Q_n(A, W)$.

We can obtain simple QPT only for several special choices of W. Therefore, to take a special value for ϕ as above tends to make the strip singular. A necessary condition for the regular case to be realised in $Q_n(A, W)$ is that $Q_n(W)$ has a local configuration consistent with the point symmetry group $\mathrm{G}(A)$ because, in the regular case, the local configuration of $Q_n(A, W)$ around the origin is required to have this property. It is found that a regular case is realised in $Q_8(\Gamma, \Pi'_8)$, $Q_{10}(P, \Pi_{10})$, $Q_{10}(P', \Pi_{10})$, $Q_{10}(P, \Sigma_{10})$, $Q_{12}(\Gamma, \Pi'_{12})$ etc, while a singular case is in $Q_8(O, \Pi'_8)$, $Q_{10}(\Gamma, \Pi_{10})$, $Q_{12}(\Gamma, \Pi_{12})$ etc.

Thus, the classification of the special points of L_n made in §§ 3 and 4 is considered to be a classification of n-gonal quasilattices with global point symmetries.

A global point symmetry of a quasilattice is unchanged by a self-similarity transformation. However, the class of the relevant special point at the origin can be changed. For example, $Q_{10}(P, \Pi_{10})$ and $Q_{10}(P', \Pi_{10})$ are interchanged by a self-similarity transformation with $\tau = (1+\sqrt{5})/2$. That is, the central tile in the former is a pentagon but that of the latter is a pentagonal star (see figure 1 in Niizeki 1989a). It is a general feature that different members of a multiplet are permuted cyclically under a self-similarity transformation of an n-gonal quasilattice.

The theory in this section is a generalisation of that developed by de Bruijin (1981) for the case of the Penrose lattice.

8. Discussion

From the discussion in § 6, we may say that the definition of the special points of a quasilattice has some arbitrariness, contrary to the case of a periodic lattice. Let A be a class of special points of L_n. Then, every point in $\pi(L_n[A])$ has the potential to be a special point of $Q_n(W)$. Whether a point in $\pi(L_n[A])$ is a special point of $Q_n(W)$ or not is determined by some criterion from the local configuration around it. However, there are no *a priori* criteria. Therefore, we may choose an criterion appropriate to the purpose of the use of the special points.

For example, $Q_8(\Pi'_8)$ (or $Q_{12}(\Pi'_{12})$) has vertices of local symmetry of D_8 (or D_{12}), i.e. the full symmetry, but $Q_{10}(\Pi_{10})$, $Q_{10}(\Sigma_{10})$ or $Q_{12}(\Pi_{12})$ has no vertices of the full symmetry. Whether we should consider all the lattice points of $Q_n(W)$ to be Γ points, as assumed in § 6, or whether we should limit to the vertices of the full point symmetry cannot be determined by *a priori* means.

The theory developed in § 3 is quite general and applicable to a classification of special points of an n-gonal lattice with $n \geq 14$ provided that $\boldsymbol{Z}(\zeta)$ is a principal ideal ring.

The present theory will be a starting point for classifying the special points of an n-gonal lattice with a non-symmorphic space group. It is, on the other hand, straightforward to extend the present theory to the classification of 'special lines' or 'special planes' of an n-gonal lattice.

It can be shown easily that L_n^*, the reciprocal lattice of L_n, is also an n-gonal lattice. Therefore, the classification of the special points of L_n^* has been completed by the present work. A detailed argument on the special points in the reciprocal space of an n-gonal quasilattice will be published elsewhere.

Most of the results of the present paper can be extended to the icosahedral quasilattices in 3D. There are three kinds of Bravais-type icosahedral quasilattices. A complete classification of their special points is published in the following paper.

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